Schurman RNDINT Model - Extracting The Risk-Neutral Default Intensity From The Price Of A Risky Coupon Bond

Gary Schurman MBE, CFA

June, 2016

There are times when pricing bonds that we want to switch from the actual probability distribution to the risk-neutral probability distribution. When working with actual probabilities we use the actual default intensity. When working with risk-neutral probabilities we use the risk-neutral default intensity. In this white paper we will develop a model for extracting the default intensity from the price of a risky coupon bond. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with pricing a non-traded risky coupon bond. We have determined that our bond has the same risk profile as a traded bond issued by ABC Corporation. The table below presents the terms and risk profile of the ABC corporate bond...

Table 1 - ABC Corporate Bond

Description	Value	Notes
Bond face value	1,000	Dollars paid at maturity given no default
Bond term	4.00	Term in years
Yield to maturity	7.00	Annual percentage rate
Risk-free rate	3.00	Annual percentage rate
Coupon rate	5.00	Annual percentage rate
Coupon payment frequency	2	Number of coupon payments per year
Default recovery rate	40	Percentage of principal recovered given default

Question 1: What is the market price of this bond at time zero?

Question 2: What is the risk-neutral default intensity?

The Traditional Bond Pricing Equation

With traditional bond pricing the price of a risky bond at time zero equals the present value of the contractual pre-default cash flows discounted at the bond's yield to maturity. To make the mathematics more tractable we will use the continuous time equivalents of discrete time rates. Using Table 1 above the table below presents the bond pricing parameters for our problem...

Table 2 - ABC Bond Pricing Parameters

Symbol	Value	Description	Notes
B_0	1,000	Bond face value	Principal paid at maturity
N	2	Number of coupon payments per year	
T	4.00	Bond term in years	
α	0.0296	Risk-free rate (continuous time)	$\ln(1.00 + 0.03)$
Δ	0.0494	Coupon rate (continuous time)	Use Equation (1) below
κ	0.0688	Yield to maturity (continuous time)	Use Equation (1) below
ϕ	0.4000	Default recovery rate	

We want to work in continuous time so we will convert the discrete time bond coupon rate and yield to maturity in Table 1 above to their continous time equivalents via the following equation...

$$\text{Continuous time rate} = \ln\left(1+\frac{\text{Stated Rate}}{N}\right)\times N \tag{1}$$

Using the parameters in Table 2 above the equation for bond price is...

$$P_0 = B_0\,\text{Exp}\left\{-\kappa\,T\right\} + \int\limits_0^T \Delta B_0\,\text{Exp}\left\{-\kappa\,t\right\}\delta t = B_0\left(\text{Exp}\left\{-\kappa\,T\right\} + \Delta\int\limits_0^T \text{Exp}\left\{-\kappa\,t\right\}\delta t\right) \tag{2}$$

Using Appendix Equation (26) below we can rewrite bond price Equation (2) above as...

$$P_0 = B_0\left(\text{Exp}\left\{-\kappa\,T\right\} + \frac{\Delta}{\kappa}\left(1-\text{Exp}\left\{-\kappa\,T\right\}\right)\right) \tag{3}$$

As noted above traditional bond pricing present values bond cash flows scheduled to be received (i.e. contractual cash flow) over the time interval $[0,T]$ at the bond's yield to maturity. We will define the variable C_t to be bond cash flow received over the time interval $[t,t+\delta t]$. We will further define bond cash flow as being inclusive of bond principal defaults but exclusive of bond principal prepayments. Using this definition the equation for expected bond cash flow under the risk-neutral measure Q is...

$$\mathbb{E}^Q\left[C_t\right] = \text{Bond principal payment} + \text{Bond coupon payment} + \text{Bond default recovery payment} \tag{4}$$

Using Equation (4) above we can write the equation for the price of a risky bond at time zero to be...

$$P_0 = \int\limits_0^T \mathbb{E}^Q\left[C_t\right]\text{Exp}\left\{-\alpha\,t\right\}\delta t \;\;\text{...where...}\;\; \alpha = \text{Risk-free rate} \tag{5}$$

Expected Bond Principal Balance Outstanding

We will define the variable λ to be the risk-neutral default intensity. If we exclude bond principal prepayments then the differential equation that defines the change in bond principal balance outstanding at time $0 \le t < T$ is...

$$\delta B_t = -\lambda\,B_t\,\delta t \tag{6}$$

The equation for expected bond principal balance outstanding at time t, which is the solution to differential Equation (6) above, is...

$$\mathbb{E}^Q\left[B_t\right] = B_0\,\text{Exp}\left\{-\lambda\,t\right\} \tag{7}$$

Expected Bond Coupon Payments Received

We will define the variable X_t to be the present value of bond coupon payments received over the infinitesimally small time interval $[t,t+\delta t]$ and the variable Δ to be the continuous time bond coupon payment rate. Using Equation (7) above the equation for expected coupon payments received at time t under the risk-neutral measure Q is...

$$\mathbb{E}^Q\left[X_t\right] = \Delta\,B_t\,\text{Exp}\left\{-\alpha\,t\right\}\delta t = \Delta\,B_0\,\text{Exp}\left\{-\lambda\,t\right\}\text{Exp}\left\{-\alpha\,t\right\}\delta t = \Delta\,B_0\,\text{Exp}\left\{-\left(\alpha+\lambda\right)t\right\}\delta t \tag{8}$$

We will define the variable $X_{0,T}$ to be the present value of bond coupon payments received over the time interval $[0,T]$ under the risk-neutral measure Q is...

$$\mathbb{E}^Q\left[X_{0,T}\right] = \int\limits_0^T \mathbb{E}^Q\left[X_t\right]\delta t = \int\limits_0^T \Delta\,B_0\,\text{Exp}\left\{-\left(\alpha+\lambda\right)t\right\}\delta t = \Delta\,B_0\int\limits_0^T \text{Exp}\left\{-\left(\alpha+\lambda\right)t\right\}\delta t \tag{9}$$

Using Appendix Equation (26) below the solution to Equation (9) above is...

$$\mathbb{E}^Q\left[X_{0,T}\right] = \Delta\,B_0\left(1-\text{Exp}\left\{-\left(\alpha+\lambda\right)T\right\}\right)\left(\alpha+\lambda\right)^{-1} \tag{10}$$

PV Expected Bond Default Recoveries

We will define the variable Y_t to be the present value of bond default principal recovery payments received over the infinitesimally small time interval $[t, t+\delta t]$ and the variable ϕ to be the default recovery rate. Using Equation (7) above the equation for expected default recovery payments received at time t under the risk-neutral measure Q is...

$$\mathbb{E}^Q\bigg[Y_t\bigg] = \lambda\,\phi\,B_t\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\,\delta t = \lambda\,\phi\,B_0\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,\mathrm{Exp}\bigg\{-\alpha\,t\bigg\}\,\delta t = \lambda\,\phi\,B_0\,\mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)t\bigg\}\,\delta t \tag{11}$$

We will define the variable $Y_{0,T}$ to be the present value of bond default recovery payments received over the time interval $[0, T]$ under the risk-neutral measure Q is...

$$\mathbb{E}^Q\bigg[Y_{0,T}\bigg] = \int_0^T \mathbb{E}^Q\bigg[Y_t\bigg]\,\delta t = \int_0^T \lambda\,\phi\,B_0\,\mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)t\bigg\}\,\delta t = \lambda\,\phi\,B_0\int_0^T \mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)t\bigg\}\,\delta t \tag{12}$$

Using Appendix Equation (26) below the solution to Equation (12) above is...

$$\mathbb{E}^Q\bigg[Y_{0,T}\bigg] = \lambda\,\phi\,B_0\bigg(1-\mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)T\bigg\}\bigg)\bigg(\alpha+\lambda\bigg)^{-1} \tag{13}$$

PV Expected Bond Principal Payments

We will define the variable Z_t to be the present value of bond principal payments received over the infinitesimally small time interval $[t, t+\delta t]$. Using Equation (7) above the equation for expected bond principal payments received at time t under the risk-neutral measure Q is...

$$\mathbb{E}^Q\bigg[Z_T\bigg] = B_T\,\mathrm{Exp}\bigg\{-\alpha\,T\bigg\} = B_0\,\mathrm{Exp}\bigg\{-\lambda\,T\bigg\}\,\mathrm{Exp}\bigg\{-\alpha\,T\bigg\} = B_0\,\mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)T\bigg\} \tag{14}$$

PV Expected Bond Cash Flow

Using Equations (10), (13) and (14) above the equation for bond price at time t under the risk-neutral measure Q is...

$$\begin{aligned} P_0 &= \mathbb{E}^Q\bigg[X_{0,T}\bigg] + \mathbb{E}^Q\bigg[Y_{0,T}\bigg] + \mathbb{E}^Q\bigg[Z_T\bigg] \\ &= \bigg(\Delta+\lambda\,\phi\bigg)B_0\bigg(1-\mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)T\bigg\}\bigg)\bigg(\alpha+\lambda\bigg)^{-1} + B_0\,\mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)T\bigg\} \\ &= \bigg(\Delta+\lambda\,\phi\bigg)B_0\Bigg[\bigg(1-\mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)T\bigg\}\bigg)\bigg(\alpha+\lambda\bigg)^{-1} + \mathrm{Exp}\bigg\{-\bigg(\alpha+\lambda\bigg)T\bigg\}\Bigg] \end{aligned} \tag{15}$$

Solve for Lambda

We will define the variable θ_1 as follows...

$$\theta_1 = \Delta + \lambda\,\phi \;\;\text{...where...}\;\; \frac{\delta\theta_1}{\delta\lambda} = \phi \tag{16}$$

We will define the variable θ_2 as follows...

$$\theta_2 = \alpha + \lambda \;\;\text{...where...}\;\; \frac{\delta\theta_2}{\delta\lambda} = 1 \tag{17}$$

We will define the variable θ_3 as follows...

$$\theta_3 = -T\,\theta_2 \;\;\text{...where...}\;\; \frac{\delta\theta_3}{\delta\lambda} = \frac{\delta\theta_3}{\delta\theta_2}\,\frac{\delta\theta_2}{\delta\lambda} = -T\times 1 = -T \tag{18}$$

We will define the variable θ_4 as follows...

$$\theta_4 = \mathrm{Exp}\bigg\{\theta_3\bigg\} \;\;\text{...where...}\;\; \frac{\delta\theta_4}{\delta\lambda} = \frac{\delta\theta_4}{\delta\theta_3}\,\frac{\delta\theta_3}{\delta\lambda} = \mathrm{Exp}\bigg\{\theta_3\bigg\}\times -T = -T\,\mathrm{Exp}\bigg\{\theta_3\bigg\} \tag{19}$$

We will define the variable θ_5 as follows...

$$\theta_5 = \theta_2^{-1} \text{ ...where... } \frac{\delta\theta_5}{\delta\lambda} = \frac{\delta\theta_5}{\delta\theta_2}\frac{\delta\theta_2}{\delta\lambda} = -\theta_2^{-2} \times 1 = -\theta_2^{-2} \tag{20}$$

Using the definitions in Equations (16) to (20) above we can rewrite bond price Equation (15) as follows...

$$f(\lambda) = P_0 = B_0\Bigg(\theta_1\theta_5 - \theta_1\theta_4\theta_5 + \theta_4\Bigg) \tag{21}$$

The derivative of Equation (21) above with respect to λ is...

$$f'(\lambda) = \frac{\delta P_0}{\delta\lambda} = B_0\Bigg(\frac{\delta\theta_1}{\delta\lambda}\,\theta_5 + \frac{\delta\theta_5}{\delta\lambda}\,\theta_1 - \frac{\delta\theta_1}{\delta\lambda}\,\theta_4\,\theta_5 - \frac{\delta\theta_4}{\delta\lambda}\,\theta_1\,\theta_5 - \frac{\delta\theta_5}{\delta\lambda}\,\theta_1\,\theta_4 + \frac{\delta\theta_4}{\delta\lambda}\Bigg) \tag{22}$$

To solve for the risk-neutral default intensity (λ) we use the Newton-Raphson method of solving non-linear equations.

$$\hat{\lambda} + \frac{f(\lambda) - f(\hat{\lambda})}{f'(\hat{\lambda})} = \lambda + \epsilon \tag{23}$$

Solution To Our Hypothetical Problem

The solutions to the two questions from our hypothetical problem above are...

Question 1: What is the market price of this bond at time zero?

Using bond price Equation (15) above and the data from Tables 1 and 2 above the market value of our bond at time zero is...

$$P_0 = 932.10 \tag{24}$$

Question 2: What is the risk-neutral default intensity?

Using bond price from Equation (24) above, bond price and derivatives from Equations (21) and (22) above, if we iterate Equation (23) the risk-neutral default intensity for our bond is...

$$\lambda = 0.0672 \tag{25}$$

Appendix Equations

A. Solution to the integral...

$$\begin{aligned}\int\limits_0^T \text{Exp}\Bigg\{-\mu\,t\Bigg\}\,\delta t &= -\frac{1}{\mu}\,\text{Exp}\Bigg\{-\mu\,t\Bigg\}\Bigg[_{t=0}^{t=T}\\ &= -\frac{1}{\mu}\Bigg(\text{Exp}\Bigg\{-\mu\,T\Bigg\} - 1\Bigg)\\ &= \frac{1}{\mu}\Bigg(1 - \text{Exp}\Bigg\{-\mu\,T\Bigg\}\Bigg)\end{aligned} \tag{26}$$